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ON THE CAUCHY PROBLEM IN THE MAXWELL-CHERN-SIMONS-HIGGS SYSTEM

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ABSTRACT. In this paper we shall prove the global existence of solutions of the classical Maxwell-Chern-Simons-Higgs equations in $(2 + 1)$ -dimensional Minkowski spacetime in the temporal gauge. We also prove that the topological solution of the Maxwell-Chern-Simons-Higgs system converges to that of Maxwell-Higgs system, as κ goes to zero. Thus we reproduce the classical result by Moncrief [6] on the global existence of the Maxwell-Klein-Gordon system in $(2 + 1)$ dimension.

1. INTRODUCTION AND MAIN RESULTS

We are concerned on the global existence problem for the Maxwell-Chern-Simons-Higgs model in $(2 + 1)$ -spacetime which was introduced to consider a self-dual system having both Maxwell and Chern-Simons terms [1]. The Lagrangian is

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho - \langle D_\mu\phi, D^\mu\phi \rangle - \frac{1}{2}\partial_\mu N\partial^\mu N \\ & - \frac{1}{2}(e|\phi|^2 + \kappa N - ev^2)^2 - e^2N^2|\phi|^2, \end{aligned} \quad (1.1)$$

where $g_{\mu\nu} = \text{diag}(1, -1, -1)$, ϕ is a complex scalar field, N is a real scalar field, $A = (A_0, A_1, A_2)$ is a vector field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu - ieA_j$, e is the charge of the electron, and κ is a coupling constant for the Chern-Simons term.

The Euler-Lagrange equations via variation of the action taken with respect to (A, ϕ, N) are

$$\begin{aligned} \partial_\lambda F^{\lambda\rho} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu} + 2e\text{Im}(\phi\overline{D^\rho\phi}) &= 0, \\ D_\mu D^\mu\phi + U_\phi(|\phi|^2, N) &= 0, \\ \partial_\mu\partial^\mu N + U_N &= 0. \end{aligned} \quad (1.2)$$

Letting $\rho = 0$ in (1.2), we obtain the Gauss-Law constraint

$$\partial_j F_{j0} - \kappa F_{12} - 2e\text{Im}(\phi\overline{D_0\phi}) = 0. \quad (1.3)$$

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The static energy functional for the system is

$$E = \int_{\mathbf{R}^2} \frac{1}{2} F_{0i}^2 + \frac{1}{2} F_{12}^2 + |D_\mu \phi|^2 + |\partial_\mu N|^2 + U(|\phi|^2, N), \quad (1.4)$$

where $U(|\phi|^2, N) = \frac{1}{2}(e|\phi|^2 + \kappa N - ev^2)^2 + e^2 N^2 |\phi|^2$, ($i = 1, 2, \mu = 0, 1, 2$). We note that, if (A, ϕ, N) is a solution that makes E finite, one of the following conditions should be required;

$$\phi \rightarrow 0 \quad \text{and } N \rightarrow \frac{ev^2}{\kappa} \quad (\text{non-topological}) \quad (1.5)$$

$$|\phi|^2 \rightarrow v^2 \quad \text{and } N \rightarrow 0 \quad (\text{topological}). \quad (1.6)$$

The terms of non-topological solution refers to the solution satisfying (1.5) and topological solution to the solution satisfying (1.6). [1], [2]

In the static case, above system are reduced to the system of elliptic equation. The static energy functional is

$$E = \int |(D_1 \pm iD_2)\phi|^2 + |D_0\phi \mp ie\phi N|^2 \\ \frac{1}{2} |F_{12} \pm (e|\phi|^2 + \kappa N - e)|^2 do \pm e \int F_{12} do.$$

The solution saturating a lower bound for the energy is called self-dual solution, which studied extensively on both two conditions (1.5), (1.6) by D. Chae *et al.* ([2], [4]), and on a periodic boundary condition, by Tarantello [5]. They also studied the unifying feature of Maxwell-Chern-Simons-Higgs mathematically, which was formally discribed in [1].

For a time dependent solution to the Maxwell-Chern-Simons-Higgs, there is no result as we know, however, in [6], Mongrief proved the global existence for the classical Maxwell-Klein-Gordon equations using the Lorents gauge in (2+1) spacetime. The Lagrangian of the Maxwell-Klein-Gordon is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \langle D_\mu \phi, D^\mu \phi \rangle.$$

He proved global existence by showing that a suitably defined higher order energy, though not strictly conserved, does not blow up in a finite time. In this article, we consider the global existence of the classical Maxwell-Chern-Simons-Higgs in the temporal gauge as well as a convergent result as the static case [3].

Before presenting main theorems, we state equations corresponding to the non-topological case in the temporal gauge.

Considering the non-topological solution of (1.2), (1.3), we put \tilde{N} to

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$N - \frac{ev^2}{\kappa}$ in (1.2) to obtain the following system of semilinear wave equations with constraint ($\square = \partial_{tt} - \Delta$).

$$\square A_1 = -\kappa \partial_0 A_2 + 2eIm(\phi \overline{D_1 \phi}), \quad (1.7)$$

$$\square A_2 = \kappa \partial_0 A_1 + 2eIm(\phi \overline{D_2 \phi}),$$

$$\square \phi = -ie\phi \partial_j A_j - 2ieA_j \partial_j \phi - e^2 A_j^2 \phi - U_{\overline{\phi}}$$

$$\square N = -U_{\overline{N}},$$

$$\partial_j F_{j0} - \kappa F_{12} - 2eIm(\phi \overline{\psi_0}) = 0. \quad (1.8)$$

Above equations can be rewritten as Hamiltonian formalism;

$$\partial_0 A_j = F_{0j}$$

$$\partial_0 F_{0j} = -\epsilon^{jk} \partial_k F_{12} - \kappa \epsilon^{jk} F_{0k} - 2eIm(\phi \overline{\psi_j})$$

$$\partial_0 F_{12} = \epsilon^{ij} \partial_i F_{0j}$$

$$\partial_0 \phi = \psi_0$$

$$\partial_0 \psi_0 = D_j \psi_j - U_{\overline{\psi}} \quad (1.9)$$

$$\partial_0 \psi_j = D_j \psi_0 - ieF_{0j} \phi$$

$$\partial_0 N = \Omega_0$$

$$\partial_0 \Omega_0 = \partial_j \Omega_j - U_N$$

$$\partial_0 \Omega_j = \partial_j \Omega_0$$

supplemented by constrains,

$$F_{jk} = \partial_j A_k - \partial_k A_j$$

$$D_j \phi = \psi_j$$

$$\partial_j N = \Omega_j \quad (1.10)$$

$$\partial_j F_{j0} - \kappa F_{12} - 2eIm(\phi \overline{\psi_0}) = 0.$$

For the topological solution we also have the equations corresponding to (1.7), (1.9) by introducing a new variable φ such that $\varphi + \lambda = \phi$ to give a natural boundary conditions to (1.2). Let us remark on some notations. If no confuses are arisen, u means a triple (A, ϕ, \tilde{N}) or (A, φ, N) ,

$$\|u(t, \cdot)\|_{H^s} = \|A(t, \cdot)\|_{H^s} + \|\phi(t, \cdot)\|_{H^s} + \|N(t, \cdot)\|_{H^s},$$

$$\|\partial_0 u(t, \cdot)\|_{H^s} = \|\partial A(t, \cdot)\|_{H^s} + \|\partial \phi(t, \cdot)\|_{H^s} + \|\partial N(t, \cdot)\|_{H^s},$$

$$\|u(t, \cdot)\|_{H^s \times H^{s-1}} = \|u(t, \cdot)\|_{H^s} + \|\partial_0 u(t, \cdot)\|_{H^{s-1}}.$$

Followings are our main theorems.

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Theorem 1.1. (*Global smooth solutions*) Consider Maxwell-Chern-Simons-Higgs. Then any finite energy H^s initial data set ($s \geq 2$) admits a unique, global solution in the temporal gauge.

$$A, \phi, \tilde{N} \in C([0, \infty); H^s(R^2)) \cap C^1([0, \infty); H^{s-1}(R^2))$$

in the non-topological case. Also in the same gauge, any finite energy H^s initial data set ($s \geq 2$) admits a unique, global solution

$$A, \varphi, N \in C([0, \infty); H^s(R^2)) \cap C^1([0, \infty); H^{s-1}(R^2))$$

in the topological case.

Theorem 1.2. (*Maxwell-Higgs Limit*) Consider the topological case of Maxwell-Chern-Simons-Higgs. Let u_κ be the global solution with coupling constant κ of H^s ($s \geq 2$) initial data u_0 . Then $\|u_\kappa(t) - u(t)\|_{H^s} \rightarrow 0$ as $\kappa \rightarrow 0$. In the case of $\kappa = 0$, if we set N initially zero then $N(t) = 0$ for all t .

Remark 1. In a succeeding section, we present the proof of the non-topological case only in Theorem 1.1 since the finite energy solution of the topological one can be found in the same way as non-topological case.

2. OUTLINE OF THE PROOFS

i) local in time existence

Proposition 2.1. Given a data set $(A, \phi, \tilde{N}) \in H^s$ ($s \geq 2$) at $t = 0$, there exists T^* depending only on $\|(A, \phi, \tilde{N})(0, \cdot)\|_{H^s}$ and a unique development (A, ϕ, \tilde{N}) in the temporal gauge with

$$(A, \phi, \tilde{N}) \in C([0, T^*]; H^s(R^2)) \cap C^1([0, T^*]; H^{s-1}(R^2)).$$

This solution can be continued as long as $\|(A, \phi, \tilde{N})\|_{H^s}(t)$ remains bounded.

First we show that there exists T^* such that (1.7) has a unique solution in X_T ,

$$X_T = \{(u, \partial_0 u) \in C([0, T^*]; H^2 \times H^1) : \|u\|_{X_T} < \infty\},$$

where $\|u\|_{X_T} = \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^2 \times H^1}$. The solution is obtained by standard contraction argument using energy estimates, and has continuous dependence on the initial data. This solution can be continued as long as $\|u_n(t, \cdot)\|_{H^2 \times H^1}$ remains bounded. To complete the local existence of Maxwell-Chern-Simons-Higgs, we also show that the constraint (1.8) is preserved in time.

ii) global in time existence

The proofs follow Mongrief's method mentioned earlier and use usual

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a priori estimates to show $\|u(t, \cdot)\|_{H^2 \times H^1}$ dose not blow up in a finite time.

Let $u = (A, \phi, N) \in C([0, T]; H^s(R^2)) \cap C^1([0, T]; H^{s-1}(R^2))$ be a solution of Maxwell-Chern-Simons-Higgs obtained in the part i). Define $E(t), E_1(t), F(t)$ as such

$$E(t) = \int_{\mathbb{R}^2} \frac{1}{2} F_{0i}^2 + \frac{1}{2} F_{12}^2 + |D_\mu \phi|^2 + |\partial_\mu N|^2 + U(|\phi|^2, N), \quad (2.1)$$

$$E_1(t) = \int_{\mathbb{R}^2} \frac{1}{2} (\partial_t F_{0i})^2 + \frac{1}{2} (\partial_t F_{12})^2 + |D_t \psi_\mu|^2 + (\partial_t \Omega_\mu)^2, \\ \text{where } \psi_\mu = D_\mu \phi, \quad \Omega_\mu = \partial_\mu N, \quad (2.2)$$

$$F_1(t) = \|\partial_i A\|_{L^2}(t) + \|\partial_i \phi\|_{L^2}(t) + \|\partial_i N\|_{L^2}(t). \quad (2.3)$$

The global result will be established after following Lemmas.

Lemma 2.2. *Let $u = (A, \phi, \tilde{N}) \in C([0, T]; H^2(R^2)) \cap C^1([0, T]; H^1(R^2))$ be a solution of (1.7), (1.8). Then*

- (1) $E(t) = E(0)$ for all $t \in [0, T]$
- (2) (A, ϕ, \tilde{N}) are estimated in L^2 in terms of the initial data for all $t \in [0, T]$;

$$\|u(t, \cdot)\|_{L^2} \leq \|u(0, \cdot)\|_{L^2} + tE. \quad (2.4)$$

Lemma 2.3. (1) $E_1(t)$ is differentiable for all $t \in [0, T]$ and satisfies

$$\begin{aligned} \partial_0 E_1(t) = & \int_{\mathbb{R}^2} -\kappa \epsilon^{ik} \partial_l F_{0i} \partial_l F_{0k} - 2e \operatorname{Im}(\psi_l \bar{\psi}_i + \phi \overline{D_l \psi_i}) \\ & + 2 \operatorname{Re}(\overline{D_l \psi_0} \cdot ie F_{lj} \psi_j - D_l U_{\bar{\phi}} + ie F_{0l} \psi_0) \\ & + 2 \operatorname{Re}(\overline{D_l \psi_j} \cdot ie F_{lj} \psi_0 - D_l (F_{0j} \phi) + ie F_{0l} \psi_j) - 2 \partial_l U_N \partial_l \Omega_0. \end{aligned} \quad (2.5)$$

- (2) $E_1(t), F_1(t)$ are estimated in terms of the initial data for all $t \in [0, T]$,

$$E_1(t) \leq C(E, E_1(0))(1+t)^2, \quad (2.6)$$

$$F(t) \leq C(E, E_1(0), F(0))(1+t)^{\frac{5}{2}}. \quad (2.7)$$

It is not clear the energy norm, $\|u(t, \cdot)\|_{H^1}$ dose not blow up in the temporal gauge, though the energy itself is preserved in Lemma 2.2. In Lemma 2.3, $E_1(t)$, the higher order energy, is shown to be initially bounded, from which $F_1(t) = \|\partial u(t, \cdot)\|_{H^1}$ can be easily estimated in terms of the initial data. Combining (2.4), (2.7) we have $\|u(t, \cdot)\|_{H^1}$ is initially bounded.

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Proving two Lemmas we use the Hamiltonian formalism of this system (1.9) after taking time derivatives in $E(t), E_1(t), F(t)$. We depend on the covariant Sobolev inequalities in [7] (see Appendix of it), estimating each terms of $\partial_0 E_1(t)$ to show the right hand terms of (2.5) are at most linear with right to $E_1(t)$. Next we introduce Brezis-Gallouet inequality.

Lemma 2.4. [8] $s > 1$,

$$\|u\|_{L^\infty} \leq C\|u\|_{H^1}(1 + \sqrt{\log(1 + \|u\|_{H^s})}).$$

Finally we carried out a priori estimate $\|u(t, \cdot)\|_{H^2 \times H^1}$ to get

$$\begin{aligned} \|u\|_{H^2}(t) + \|\partial_0 u\|_{H^1}(t) &\leq \|u\|_{H^2}(0) + C \int_0^t \|\partial_0 u\|_{H^1} \\ &\quad + ((1 + \|u\|_{L^\infty})^2 + \|u\|_{H^1}^{\frac{1}{2}}(1 + \|u\|_{L^2}))\|u\|_{H^1} \\ &\quad + \|u\|_{L^\infty}\|u\|_{H^1}^2\|u\|_{H^2}^{\frac{1}{2}} + (1 + \|u\|_{L^\infty})\|u\|_{H^1}^2 \end{aligned} \quad (2.8)$$

by energy estimates to (1.7), then we have

$$\|u(0, \cdot)\|_{H^2} + C(t) \int_0^t \log(1 + \|u(s, \cdot)\|_{H^2})\|u(s, \cdot)\|_{H^2 \times H^1} ds. \quad (2.9)$$

applying above Brezis-Gallouet inequality. The desired result, thus, is given by the general Gronwall inequality.

For the case of an initial data $u \in H^s(\mathbf{R}^2)$, it is easy to obtain a local existence result as proposition 2.1. For a global result we state a next lemma omitting its simple proof.

Lemma 2.5. *Let $(A, \phi, N) \in C([0, T]; H^s(\mathbf{R}^2)) \cap C^1([0, T]; H^{s-1}(\mathbf{R}^2))$ be a solution of (1.7), (1.8) for $s > 2$ then $\|u\|_{H^s}(t)$ is estimated in terms of the initial data for all $t \in [0, T]$.*

iii) Maxwell-Higgs limit

Let u^κ be a topological solution of Maxwell-Chern-Simons-Higgs obtained in Theorem 1.1 with coupling constant κ of H^2 initial data u_0 . It is easy to show $\|(u^\kappa - u)(t, \cdot)\|_{H^2 \times H^1}$ is estimated to be

$$\|(u^\kappa - u)(t, \cdot)\|_{H^2 \times H^1} \leq \int_0^t \kappa \|\partial_0 A_\kappa\|_{H^1} + C(t)\|u^\kappa - u\|_{H^2}, \quad (2.10)$$

using $\sup_{0 \leq \kappa \leq 1} \|u^\kappa(t, \cdot)\|_{H^2 \times H^1} \leq C(t)$ for a smooth function in the proof of Theorem 1.1. Then applying Gronwall inequality to (2.10), we have

$$\|(u^\kappa - u)(t, \cdot)\|_{H^2} \leq \kappa C(t) + \|(u^\kappa - u)(0, \cdot)\|_{H^2},$$

letting $\kappa \rightarrow 0$, we obtain the desired result.

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